Interference, diffraction, and refraction, via Dirac's notation

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The diffraction grating equation and the law of refraction are derived, in the macroscopic domain, using a generalized interference equation derived via the application of Dirac's notation to classical optics.

I. INTRODUCTION

In classical optics textbooks the diffraction grating equation and the law of refraction, also known as Snell's law, are often introduced separately.[1–4] For instance, Born and Wolf[1] introduce the law of refraction in their first chapter using Maxwell's relation, $n=\sqrt{\epsilon\mu}$. The diffraction grating equation, on the other hand, is introduced much later following their discussion of Fraunhofer and Fresnel diffraction.[1] Jenkins and White[2] introduce the law of refraction utilizing geometrical arguments very early while the diffraction grating equation is derived much later via the addition of complex amplitudes.[2] Other authors[5,6] discuss the law of refraction and the diffraction grating equation within the boundaries of geometrical optics, but in distinct sections with Snell's law being discussed first.

In this paper, a description is given on how to apply Dirac's notation,[7] in the macroscopic domain, to characterize the propagation of electromagnetic radiation via an N-slit transmission diffraction grating. In this regard, application of Dirac's notation to the relevant geometry yields a generalized interferometric equation whose interference term leads to the diffraction grating equation and subsequently to the law of refraction. This approach provides a unified description of interference, diffraction, and refraction. Following Feynman[8] the main mathematical assumption made here is to relate the probability amplitudes to classical complex wave functions.

The use of quantum techniques in the description of interference and other optical phenomena has been previously discussed by several authors.[8–11] Feynman applied the Dirac formalism to discuss the two-slit interference experiment.[8] He also used path integrals to describe diffraction through a single slit,[9] and illustrated the application of quantum electrodynamic methods to discuss interference, diffraction, and reflection in the microscopic domain.[10] Lamb[11] has recently outlined the use of the radiation field in quantum mechanics to characterize several optical phenomena of interest. Also, a recent discussion on the law of refraction in quantum mechanics, in the microscopic regime, has been published in this journal.[12]

In the area of mathematical techniques it is interesting to note that the methods of Hamiltonian optics, designed to deal with geometrical optics, have been considered to be analogous to those of quantum mechanics.[13] More specifically, it can be mentioned that Dirac introduced Hamilton's principle to quantum mechanics[7] and that Feynman adopted this principle in the formulation of his path integrals method.[9]

The discussion presented in this paper deals with the application of Dirac's notation in the macroscopic regime. This was first done to characterize N-slit interference[14] resulting from the interaction of coherent light with transmission gratings incorporating a large number of slits per mm. Since then, the approach has been successfully applied to predict near-field diffraction and generalized interference cases of practical interest.[15–18]

Although in this paper we are concerned with the interaction of an expanded laser beam with a transmission grating, it should be mentioned that conceptually we have based our description of interference on Dirac's hypothesis that each photon goes partly into each slit and that *each photon then interferes with itself*.[7,19] In this regard, interference is characterized via the mathematical interaction of probability amplitudes. Dirac did not limit the extent of his discussion to single photons but wrote about beams of light and photons associated with such beams. Of further interest is the fact that Dirac specified that each of the translational states of a photon is associated with *one of the wave functions of ordinary wave optics*.[7]

II. BACKGROUND

The experimental apparatus considered here is illustrated in Fig. 1. A laser beam is expanded in one dimension by a multiple-prism beam expander. The expanded laser beam then illuminates a generalized transmission grating and interference occurs at a screen, which is a charge-coupled device (CCD) detector array in this case.[17,18] The problem is now treated by considering the probability amplitude for propagation from the exit surface of the beam expander (s) to the detection screen (x) via the array of N slits (j). Hence, we can write[7,8,14,15]

$$\langle x|s\rangle=\sum_{j=1}^{N}\langle x|j\rangle\langle j|s\rangle. \tag{1}$$

Here, the probability amplitudes can be expressed as plane waves[8] in the form of $\langle j|s\rangle=\Psi(r_{s,j})\exp(-i\theta_j)$ and $\langle x|j\rangle=\Psi(r_{j,x})\exp(-i\phi_j)$. Here, θ_j and ϕ_j are the phase terms associated with the incidence and diffraction waves, respectively. For a time-independent plane wave these phase terms become the wave number (k) multiplied by the respective path difference.[1,8]

Equation (1) can be written as

$$\langle x|s\rangle=\sum_{j=1}^{N}\Psi(r_j)e^{-i\Omega_j}, \tag{2}$$

where $\Psi(r_j)=\Psi(r_{s,j})\Psi(r_{j,x})$ and $\Omega_j=(\theta_j+\phi_j)$. The propagation probability can be obtained by expanding Eq. (2) and multiplying the expansion with its complex conjugate. Rearranging terms and using the identity

$$2\cos(\Omega_m-\Omega_j)=e^{-i(\Omega_m-\Omega_j)}+e^{i(\Omega_m-\Omega_j)},$$

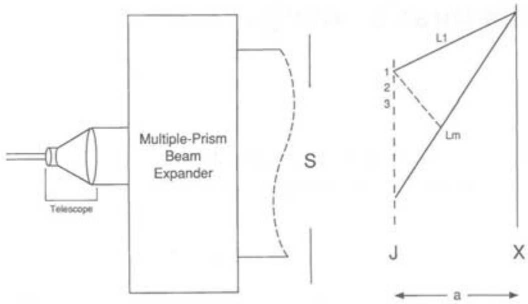


Fig. 1. A laser beam is expanded by a multiple-prism beam expander whose exit plane is denoted by (s). The expanded laser beam then illuminates a generalized grating (j). Interference occurs at the screen or CCD array labeled by (x) (from Ref. 17).

we can write the generalized propagation probability in one dimension,[14,15]

$$|\langle x|s\rangle|^2=\sum_{j=1}^{N}\Psi(r_j)^2+2\sum_{j=1}^{N}\Psi(r_j)\times\left(\sum_{m=j+1}^{N}\Psi(r_m)\cos(\Omega_m-\Omega_j)\right). \tag{3}$$

Figure 2 compares the measured interferometric radiation distribution for a 100-slit grating, at a grating-detector distance of 75 cm, with the predicted distribution using Eq. (3) for plane wave illumination of the grating.[17,18] Additional comparison cases are available in the literature.[17,18] The case of illumination via a wide slit introducing a diffraction distribution at the j plane has also been considered. Furthermore, an equation for the two-dimensional case has been given.[18]

A more detailed view of the grating plane is given in Fig. 3. The phase difference term in Eq. (3) can be written as[1]

$$\cos\{(\theta_m-\theta_j)\pm(\phi_m-\phi_j)\}=\cos\{(l_m-l_{m-1})k_1\pm(L_m-L_{m-1})k_2\}, \tag{4}$$

where $k_1=2\pi n_1/\lambda_v$ and $k_2=2\pi n_2/\lambda_v$ are the corresponding wave numbers of the optical regions defined in Fig. 3(b). Here, we have used $\lambda_1=\lambda_v/n_1$ and $\lambda_2=\lambda_v/n_2$ where λ_v is the vacuum wavelength and n_1 and n_2 are the corresponding indexes of refraction.[1,20] The linear dependence of λ_v on the refractive index, according to $\lambda_v=\lambda n$, is a well-known physical relation[1] widely applied in the wavelength tuning of laser resonators.[20]

In the phase difference terms the path differences can be expressed exactly via geometrical equations such as[17]

$$|L_m-L_{m-1}|=2\xi_m d_m/|L_m+L_{m-1}|, \tag{5}$$

$$L_m^2=a^2+(\xi_m+(d_m/2))^2, \tag{6}$$

$$L_{m-1}^2=a^2+(\xi_m-(d_m/2))^2, \tag{7}$$

where ξ_m is the lateral displacement, on the x plane, from the projected median of d_m to the interference location. From the geometry we can write

$$\sin\Phi_m=(\xi_m+(d_m/2))/L_m. \tag{8}$$

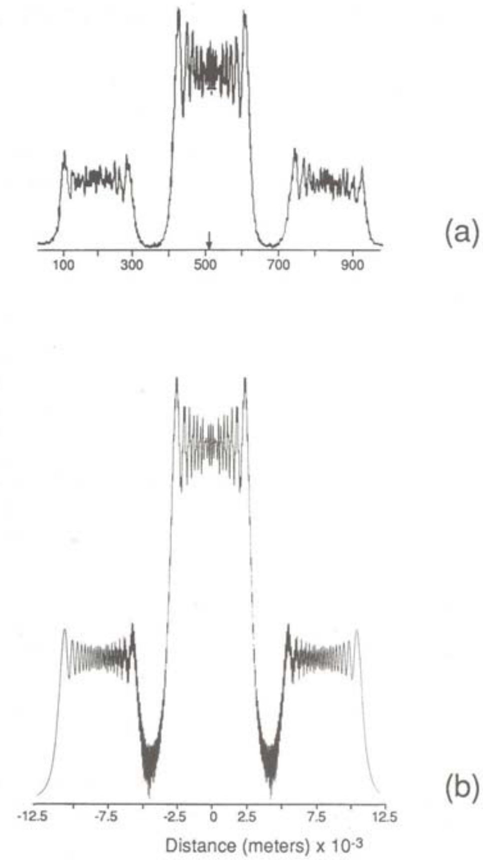


Fig. 2. Interferometric signal produced by the illumination of 100 slits 30 μm wide and separated by 30 μm. Interference at the 0th and ± 1 diffraction orders is shown. The grating–screen distance is 75 cm and the laser wavelength is 632.8 nm. The signal in (a) is a measurement. Here, the horizontal scale is given in pixels each 25 μm wide. The corresponding theoretical pattern is illustrated in (b) (from Ref. 17).

For the condition $a \gg d_m$ (see Fig. 3), we have $|L_m+L_{m-1}|\approx 2L_m$. Thus, using Eqs. (5) and (8), we can write

$$|L_m-L_{m-1}|\approx d_m\sin\Phi_m, \tag{9}$$

$$|l_m-l_{m-1}|\approx d_m\sin\Theta_m, \tag{10}$$

where Θ_m and Φ_m are the angles of incidence and diffraction, respectively.

Since maxima can occur at[1]

$$\{|l_m-l_{m-1}|n_1\pm|L_m-L_{m-1}|n_2\}(2\pi/\lambda_v)=M\pi, \tag{11}$$

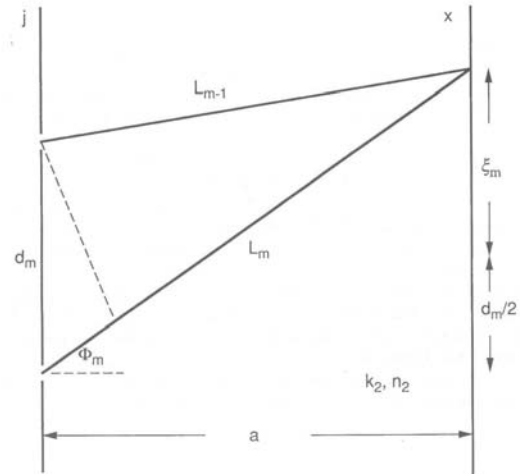
then, using Eqs. (9) and (10), we can write

$$d_m\{n_1\sin\Theta_m\pm n_2\sin\Phi_m\}(2\pi/\lambda_v)=M\pi, \tag{12}$$

where $M=0,2,4,6,\ldots$.

For $n_1=n_2=1$, we can write $\lambda=\lambda_v$ and Eq. (12) reduces to the well-known diffraction grating equation

$$d_m(\sin\Theta_m\pm\sin\Phi_m)=m\lambda, \tag{13}$$



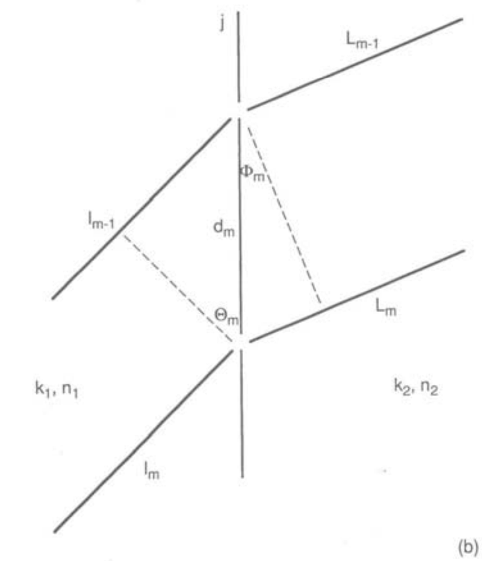


Fig. 3. (a). A detailed view of the grating plane (j) and the detection screen plane (x) depicting the geometrical parameters included in Eqs. (5)–(9). (b) The grating plane (j) for the condition $a \gg d_m$.

where $m = 0,1,2,3,\ldots$.

Now, as d_m is made very small relative to a given λ, the only possible solution for Eq. (13) is found for the case of $m=0$. For example, since the maximum value for ($\sin \Theta_m \pm \sin \Phi_m$) is 2, for a 5000-lines/mm transmission grating no diffraction can be observed for wavelengths in the visible spectrum. Hence, in this regime the transmission grating acts similarly to a glass slab. This can be easily verified by shining a HeNe laser beam ($\lambda = 632.8$ mm) on a 5000-lines/mm transmission grating.

Consequently, for a high groove density transmission grating illuminated by a sufficiently long wavelength, so that the condition $d_m \ll \lambda$ is satisfied, interference ceases to occur and only refraction is observed, thus Eq. (11) can only be solved for

$$\{|l_m - l_{m-1}|n_1 \pm |L_m - L_{m-1}|n_2\}(2\pi/\lambda_v) = 0. \tag{14}$$

Thus we can have

$$n_1 \sin \Theta_m = n_2 \sin \Phi_m . \tag{15}$$

For an air–glass interface, $n_1 \approx 1$, and we may write

$$\sin \Theta_m \approx n_2 \sin \Phi_m, \tag{16}$$

which has the form of the well-known law of refraction.

III. DISCUSSION

Interference, diffraction, and refraction phenomena have often been discussed as distinct subjects in optical textbooks.[1–4] In this paper it is suggested that these subjects can be considered in a unified approach, in the macroscopic domain, with the fundamental description for interference giving rise to the diffraction grating equation and the law of refraction.

This unified approach involves the application of Dirac's notation to a transmission grating geometry to yield a generalized interference equation. From this interference equation the phase difference term is used in conjunction with the same geometry to derive an expression for the diffraction grating equation and the law of refraction for the limiting case of $d_m \ll \lambda$. It is useful to reiterate that the basic element in this derivation is the interaction of classical wave functions according to the mathematical construct provided by the application of Dirac's notation to propagation of electromagnetic radiation from a source to a detection array via a transmission grating.

This method leads to the diffraction grating equation in a fairly straightforward manner. Deducing the law of refraction requires the introduction of the indices of refraction as mathematical constants, via $\lambda_v = \lambda n$, and the application of a physical observation in the limiting case of $d_m \ll \lambda$. Although an explanation of the origin of the refraction index would require invoking microscopic arguments,[12] beyond our scope, the Dirac method does offer a simple, brief, and unified description of fundamental optical phenomena. The generalized aspects of this method combined with the facility it offers to solve problems of practical interest[17,18] should offer appealing characteristics from pedagogical and utilitarian perspectives. Although it is most certain that Feynman foresaw the pedagogical and scientific value of the application of Dirac's notation to the two-slit experiment,[8] it is less certain whether anyone could have predicted its practical value.[17,18]

IV. CONCLUSION

An alternative unified avenue to the description of interference, diffraction, and refraction has been provided. This method applies Dirac's notation to describe electromagnetic wave propagation to a generic optics geometry involving a source, a transmission grating plane, and an interference plane. The simple mathematical formalism of the description provides a pedagogical alternative and significant practical advantages.[17,18]

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